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ON THE GENERAL THEORY OF CURVED SURFACES AND RECTILINEAR CONGRUENCES

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During the past two or three years, I have presented several communications to the American Mathematical Society concerning the general theory of curved surfaces and rectilinear congruences. I have been unable to find the leisure to write out in full my results, and take the present opportunity to gather together a few of the more important, in the hope that I may soon be in position to publish an extended treatment elsewhere.

Let the non-developable surface S be referred to a non-conjugate parametric net (u, v). Its equations in homogeneous coordinates may be taken in the form

$$y^{(k)} = y^{(k)}(u, v) \qquad (k = 1, 2, 3, 4), \tag{1}$$

where the determinant $|y_{uv}, y_u, y_v, y|$ is nowhere zero, and the four functions $y^{(k)}$ will then be a fundamental system of solutions of a completely integrable system of partial differential equations[1]

$$\begin{aligned} y_{uu} &= a\, y_{uv} + b\, y_u + c\, y_v + d\, y, \\ y_{vv} &= a' y_{uv} + b' y_u + c' y_v + d' y. \end{aligned} \tag{2}$$

The coefficients in these equations are functions of u, v which satisfy certain conditions of complete integrability, which we shall not write out here. Suffice it to say that if the integrability conditions are identically satisfied, any derivative of y is expressible, and in only one way, as a linear combination of y_{uv}, y_u, y_v, y,

$$\frac{\partial^{p+q} y}{\partial u^p \partial v^q} = a^{(pq)} y_{uv} + b^{(pq)} y_u + c^{(pq)} y_v + d^{(pq)} y. \tag{3}$$

The two points defined by the equations

$$\rho = y_u - \nu y, \qquad \sigma = y_v - \mu y, \tag{4}$$

where μ and ν are functions of u, v, lie on the tangents to the parametric curves. Let us denote by l the line joining ρ and σ; we have thus associated with each point y of the surface a definite line l lying in the corresponding tangent plane. The congruence of lines l we shall denote by Γ.

Again, if μ and ν are any functions of u, v, the point

$$z = y_{uv} - \mu y_u - \nu y_v \tag{5}$$

does not lie in the tangent plane to S at y, and the line l' joining y and z therefore protrudes from the surface. The lines l' constitute a congruence Γ'.

If, now, the functions μ and ν are the same in equations (4) and (5), the lines l and l' are in a certain characteristic geometric relation, which may be described as follows: The ruled surface $R^{(u)}$ formed by the tangents to the curves v = constant along a fixed curve u = constant is skew, since the parametric net is non-conjugate. This ruled surface is touched, in the point ρ defined by (4), by the plane determined by the line l' and the tangent $y\, y_u$. Similarly the point σ is the point in which the plane determined by l' and the other parametric tangent $y\, y_v$ is tangent to the ruled surface $R^{(v)}$ formed by the tangents to the curves u = constant along a fixed curve v = constant. The line l is therefore determined uniquely by the line l', and, conversely, if the line l is given, or in other words the points ρ and σ, the line l' is defined as the line of intersection of the tangent planes to the ruled surfaces $R^{(u)}$ and $R^{(v)}$ constructed at the points ρ and σ. The relation between the lines l and l', or between the congruences Γ and Γ', is a reciprocal one, which for want of a better name I shall call the *relation R*. It is determined of course by the particular parametric net to which the surface is referred.

The developables of the congruence Γ' cut the surface S in a net of curves whose differential equation is

$$\begin{aligned}&\left[c^{(21)} - c\mu - \nu_u + \nu\left(a^{(21)} - a\mu - \nu\right)\right] du^2 \\ &+\left\{c^{(12)} - c'\nu - \nu_v + \nu\left(a^{(12)} - a'\nu - \mu\right) - \left[b^{(21)} - b\mu - \mu_u + \mu\left(a^{(21)} - a\mu - \nu\right)\right]\right\} du\,dv \\ &-\left[b^{(12)} - b'\nu - \mu_v + \mu\left(a^{(12)} - a'\nu - \mu\right)\right] dv^2 = 0.\end{aligned} \tag{6}$$

The quantities with double upper indices are coefficients of equations of the form (3). Likewise, the developables of the congruence Γ correspond to a net of curves on S defined by the differential equation

$$[d-\nu_u+(b-\nu)\,\nu+c\mu+a\,(\mu_u+\mu\nu)\,]\,du^2$$
$$+\left\{a'[d-\nu_u+(b-\nu)\,\nu+c\mu]-a[d'-\mu_v+b'\nu+\mu\,(c'-\mu)]+\mu_u-\nu_v\right]dudv \quad (7)$$
$$-[\,d'-\mu_v+b'\nu+\mu\,(c'-\mu)+a'\;(\nu_v+\mu\nu)\,]\,dv^2=0.$$

The consideration of these developables, together with the focal points of the lines l and l', leads to many theorems which are generalizations of known theorems concerning conjugate nets. The point-conjugate of the surface S with respect to Γ', i.e., the surface generated by the harmonic conjugate of y with respect to the focal points of l', plays an important part in the discussion.

Of more immediate interest is the case in which the parametric net consists of the asymptotic curves of S. The differential equations of the surface may then be written[2]

$$y_{uu}+2\,b\,y_v+f\,y=0,\; y_{vv}+2\,a'y_u+gy=0, \quad (8)$$

and the differential equations defining the developables of the congruences Γ' and Γ respectively

$$\begin{gathered}[f+\nu^2+\nu_u-2b\mu+2b_v]\,du^2+(\nu_v-\mu_u)\,dudv\\ -[g+\mu^2+\mu_v-2a'\nu+2a^2_u]\,dv^2=0,\end{gathered} \quad (9)$$

$$\begin{gathered}[f+\nu^2+\nu_u+2b\mu]\,du^2+(\nu_v-\mu_u)\,dudv\\ -[\,g+\mu^2+\mu_v+2a'\nu]\,dv^2=0.\end{gathered} \quad (10)$$

Especially interesting in this connection are the directrix congruences defined by Wilczynski.[3] These are in the relation R, since the directrix of the first kind is the line joining the points

$$r=y_u-\frac{a'_u}{2a'}\,y, \qquad s=y_o-\frac{b_v}{2b}\,y, \quad (11)$$

and the directrix of the second kind is the line joining the point y with the point

$$t=y_{uv}-\frac{b_v}{2b}\,y_v-\frac{a'_u}{2a'}\,y_v. \quad (12)$$

The following new geometric characterization of the directrix congruences may be given: *two congruences* Γ *and* Γ' *which are in the relation* R *to the asymptotics of a surface* S, *are the directrix congruences of* S *if and only if their developables correspond to the same net on* S.

A number of propositions, which Wilczynski has proved for the directrix congruences, subsist also for any congruences Γ and Γ' in the relation R with respect to the asymptotic net of the surface.

In the general case, i.e., when the parametric net is not necessarily asymptotic, the developables of the congruence Γ correspond to a conjugate net on S if and only if $\mu_u - \nu_v = 0$. Borrowing a locution of Guichard's, used by him in a quite different connection, I shall say that a congruence Γ is *harmonic to a surface* S if its developables correspond to a conjugate net on S. *If a congruence* Γ *is harmonic to a surface* S, *or if its developables correspond to the asymptotic curves of* S, *and then a line* l *of* Γ *is met in its focal points by the tangents to the curves corresponding to the developables of* Γ, *and conversely.* This theorem is a generalization of a corresponding theorem for the case in which the parametric net is conjugate and the congruence Γ is the *ray congruence*, i.e., the congruence of lines joining the first and minus first Laplace transforms of the parametric conjugate net. The theorem affords a new geometric characterization of conjugate nets with equal Laplace-Darboux invariants.[4]

A geometric characterization of isothermal nets may be given in terms of the relation R. Let the surface be referred to an orthogonal net, and let the congruence Γ' consist of the normals to the surface. Then the parametric net is isothermal if and only if the developables of the related congruence Γ correspond to a conjugate net on S. This characterization, together with another, is soon to appear in the Transactions of the American Mathematical Society.

In what follows, I shall always suppose that the asymptotic net is parametric. Again borrowing a terminology used in a different sense by Guichard, I shall say that a congruence Γ' is *conjugate to a surface* S if its developables cut the surface in a conjugate net. Then *if the congruence* Γ' *is conjugate to the surface* S, *the related congruence* Γ *is harmonic to* S, *and conversely.* The two conjugate nets can coincide only when Γ and Γ' are the directrix congruences.

An important question naturally arises concerning the existence of congruences conjugate to a given surface and uniquely determined thereby. The surface normals form such a congruence, so that a projective generalization of metric theorems would demand the existence of a congruence projectively determined by the surface and conjugate to it. I have found that such a congruence is generated by the lines l' joining the point y with the point

$$\zeta = y_{uv} + \tfrac{1}{2}\left(\frac{a'_v}{a'} + \frac{b_v}{b}\right) y_u + \tfrac{1}{2}\left(\frac{a'_u}{a'} + \frac{b_u}{b}\right) y_v. \tag{13}$$

The corresponding line l joins the points

$$\rho = y_u + \tfrac{1}{2}\left(\frac{a'_u}{a'} + \frac{b_u}{b}\right)y, \quad \sigma = y_v + \tfrac{1}{2}\left(\frac{a'_v}{a'} + \frac{b_v}{b}\right)y. \tag{14}$$

The following geometric characterization of these points ρ and σ will of course afford a characterization of the line $y\zeta$.

Let l be any line in the tangent line, and l' the corresponding line through the point y. Project the asymptotic curves on the tangent plane, from any point on l', and let C_1 and C_2 be the conics which osculate these projections at y. There exists but one pair of lines l, l' such that the interesections of l with the asymptotic tangents are the double points of the involution determined by the two pairs of points in which l cuts the conics C_1 and C_2. Let R and S be these double points; they are given by the expressions

$$R = y_u + \tfrac{1}{4}\frac{b_u}{b}y, \quad S = y_v + \tfrac{1}{4}\frac{a'_v}{a'}y. \tag{15}$$

Recalling that the directrix of the first kind intersects the asymptotic tangents in the points r and s defined by equations (11), one finds that the point ρ is the harmonic conjugate of r with respect to y and R, and σ is the harmonic conjugate of s with respect to y and S. This completes the required characterization.

The points R and S, whose coordinates are given by equations (15), are of importance for still another reason. Darboux has shown[5] that in terms of the non-homogeneous coordinates of a regular point of a surface the equation of the surface may be written in essentially either of the following two forms, provided a local tetrahedron of reference be properly chosen:

$$z = xy + \tfrac{1}{6}(x^3 + y^3) + \tfrac{1}{24}(Ix^4 + Jy^4) + \ldots,$$
$$z = xy + \tfrac{1}{6}(x^3 + y^3) + \tfrac{1}{24}xy(I^1x^2 + J^1y^2) + \ldots.$$

Darboux did not characterize either tetrahedron. The tetrahedron which gives rise to the first expansion was completely characterized by Wilczynski.[3] To obtain the second expansion, three of the vertices of the tetrahedron of reference must be taken at the points y, R, S, and the fourth at the intersection of the canonical quadric with the line corresponding to RS in the relation R.

The congruence of lines $y\zeta$, since its developables cut the surface in a conjugate net, would very naturally take the place of the congruence of normals to a surface in projective generalizations of metric theorems. The said conjugate net would then play the part of the lines of curva-

ture. If this conjugate net has equal Laplace-Darboux invariants, a particular class of surfaces analogous to isothermic surfaces is defined. A projective generalization of geodesics may be made in terms of the congruence $y\zeta$, since[6] there exists a two-parameter family of curves on the surface whose osculating planes contain the lines $y\zeta$. It must be possible also to generalize a good part of the theory of triply orthogonal systems and families of Lamé, although the generalization can never be complete on account of the essential differences between metric and projective space. The field seems, on the whole, to be very promising.

[1] Green, G. M., *Trans. Amer. Math. Soc., New York,* **17,** 1916, (483–516).

[2] Wilczynski, E. J., *Ibid.,* **8,** 1907, (233–260).

[3] Idem, *Ibid.,* **9,** 1908, (79–120).

[4] Green, G. M., *Amer. J. Math., Baltimore,* **38,** 1916, (313).

[5] Darboux, *Bull. Sci. Math., Paris,* (Ser. 2), **4,** 1880, (348–384).

[6] Cf. the abstract of Miss P. Sperry, *Bull. Amer. Math. Soc., New York,* **22,** 1915–1916, (441–442). The normal congruence is there replaced by the directrix congruence of the second kind, whose developables, however, do not cut the surface in a conjugate net.

A CONTRIBUTION TO THE PETROGRAPHY OF SOUTHERN CELEBES

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In a paper in the *Journal of Geology,* Chicago, **23,** 1915, (231–245), the authors described some rocks collected in Java and Celebes in 1910. The chemical analyses of seven of these were from lavas and coarsely crystalline igneous rocks occurring in the neighborhood of Bulu Saraung (Pic de Maros). The rocks analyzed are trachytes, absarokite, nephelite-syenite and fergusite, besides kentallenite and marosite, rocks related to shonkinite.

In November, 1914, a more extended visit was made to Southern Celebes under the auspices of the Bureau of Mines of the Netherlands Government. The mountainous region from Maros to Malawa and Batuku was studied in company with Mr. 'T Hoen and Mr. Ziegler, geologists of the Bureau. The region visited consists of several nearly parallel ranges of volcanic mountains, whose lavas are underlaid by faulted and dislocated strata which are exposed in the valleys and along the base of the volcanic ridges.

The faulting and dislocation of the limestones and coal-bearing shales antedated the eruption of the igneous rocks, for the distorted strata are overlaid by volcanic breccias which form much less disturbed beds